NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

Further International Selection Test

14th May 1981 Time allowed - $3\frac{1}{2}$ hours

Write on one side of the paper only. Start each question on a fresh sheet of paper and arrange your answers in order. Put your full name, age (in years and months) and school on the top sheet of your answers. On each other sheet put your name and initials.

The earlier questions may be found shorter and easier; they carry fewer marks.

1. A given triangle has its three sides coloured red. Three blue straight lines cut the triangle into seven pieces, four of which are triangles and three are pentagons. Two of the sides of each triangle are blue and one of them has its third side blue also.

Given that all four triangles are congruent, express the area of each as a fraction of the area of the given triangle.

2. An axis of a solid is, for the purposes of this question, defined to be a straight line joining two points on the surface of the solid and such that the solid, when rotated about this line through an angle which is greater than 0° and less than 360°, coincides with itself.

How many axes has a cube? Draw three diagrams to show the three different types of axis and state the minimum angle of rotation for each type.

(No formal proofs are required. Each diagram should show clearly the axis, the vertices of the cube numbered from 1 to 8 and a symbol like $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 3 & 1 & 4 & 8 & 6 & 7 \end{pmatrix}$ indicating that the points 1,2,3,...8 move to the points 2,5,3,...,7 respectively.)

3. Solve for x,y,z the simultaneous equations

$$x^2y^2 + x^2z^2 = axyz, \quad y^2z^2 + y^2x^2 = bxyz, \quad z^2x^2 + z^2y^2 = cxyz$$

where a,b,c are given numbers.

4./.. TURN OVER

4. Find the remainder when the polynomial

$$x^{81} + x^{49} + x^{25} + x^9 + x$$

is divided by the polynomial $x^3 - x$.

5. The sequence $\{u_n\}$ of real numbers is defined for $n \geq 0$ by

$$u_0 = 2, \quad u_1 = 5$$

and

$$u_{n+1}u_{n-1} - u_n^2 = 6^{n-1} \quad \text{when } n \geq 1.$$

Prove that each u_n is an integer.

6. Prove that if c is a rational number the equation

$$x^3 - 3cx^2 - 3x + c = 0$$

has at most one rational root.

7. Prove that if x,y are non-negative integers then $5x \geq 7y$ if and only if there exist non-negative integers a,b,c,d such that

$$x = a + 2b + 3c + 7d,$$
$$y = \phantom{a + {}} b + 2c + 5d.$$